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from

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Power System Relaying, Theory and Applications

Know your physics and the rest will follow (R. K. Kalman)

Chapter 3
Modeling for Power System Relaying Analysis

3.1 Introduction

The power system comprises generators, step-up/step-down transformers, autotransformers, transmission lines (overhead or underground operating at various kV levels), reactors, capacitors, distribution lines, end-use equipment (customers), motors, etc. As examples, Figure 3.1 illustrates an artistic visualization of power systems and Figure 3.2 illustrates typical distribution systems (utility) and end use equipment (customers). The illustrated power system and the medium voltage distribution systems are typical designs of US utilities to supply electric power to commercial, residential and industrial customers.

Any power system analysis method must be able to model and analyze systems similar to the ones illustrated in Figures 3.1 and 3.2. The phenomena to be analyzed on these systems are numerous, i.e. power frequency, harmonics, dynamic transients, switching transients, lightning transients, in-rush current transients, etc. Each of these phenomena may include different frequency spectra. The models to be used should reproduce the response of the system to these phenomena with high fidelity. In this chapter we examine modeling techniques for various power system components that provide this capability.

The various electric power system components that must be modeled are:

- Transmission Lines
- Transformers
- Generators
- Induction Machines
- Capacitors
- Reactors
- Converters
- Adjustable speed drives
- Power supplies
- etc.

Some of the power system components are linear, i.e. they do not distort the applied voltage and current while others are distorting, i.e. they introduce distortion of the waveform, such as converters, adjustable speed drives, power supplies, transformers, etc.

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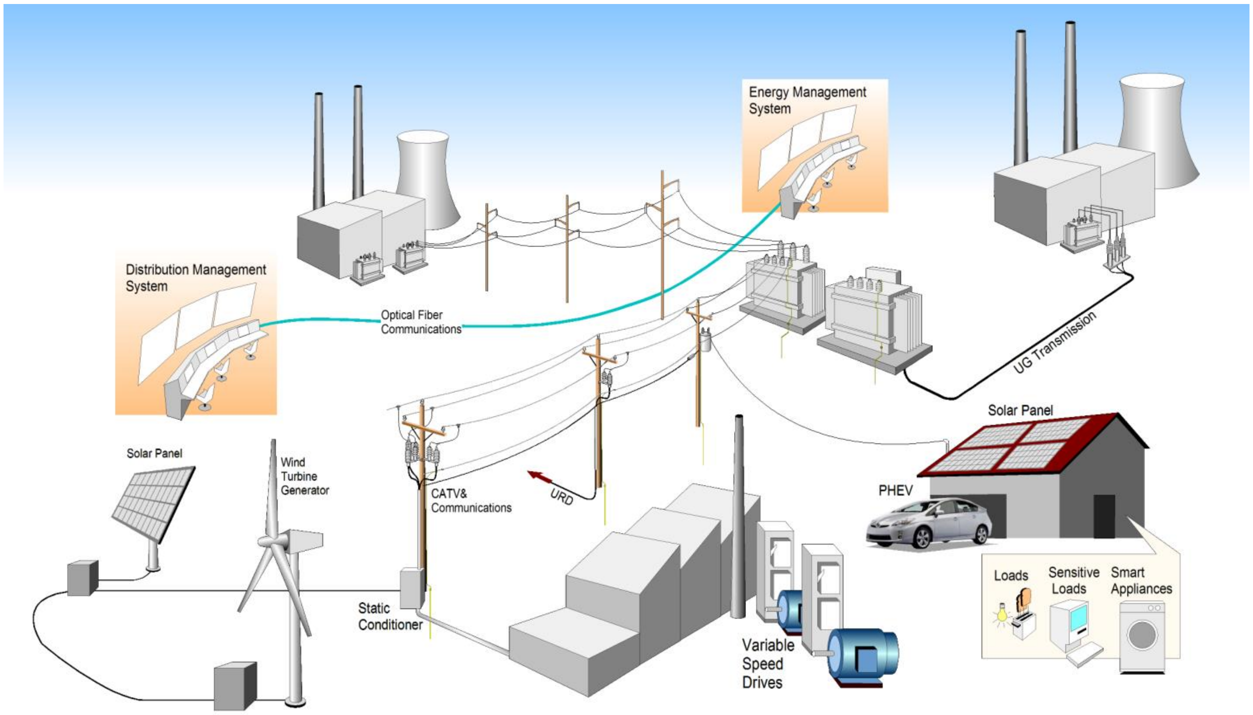

Figure 3.1: A Power System Comprising Generation, Transmission and Distribution - Overhead and Underground

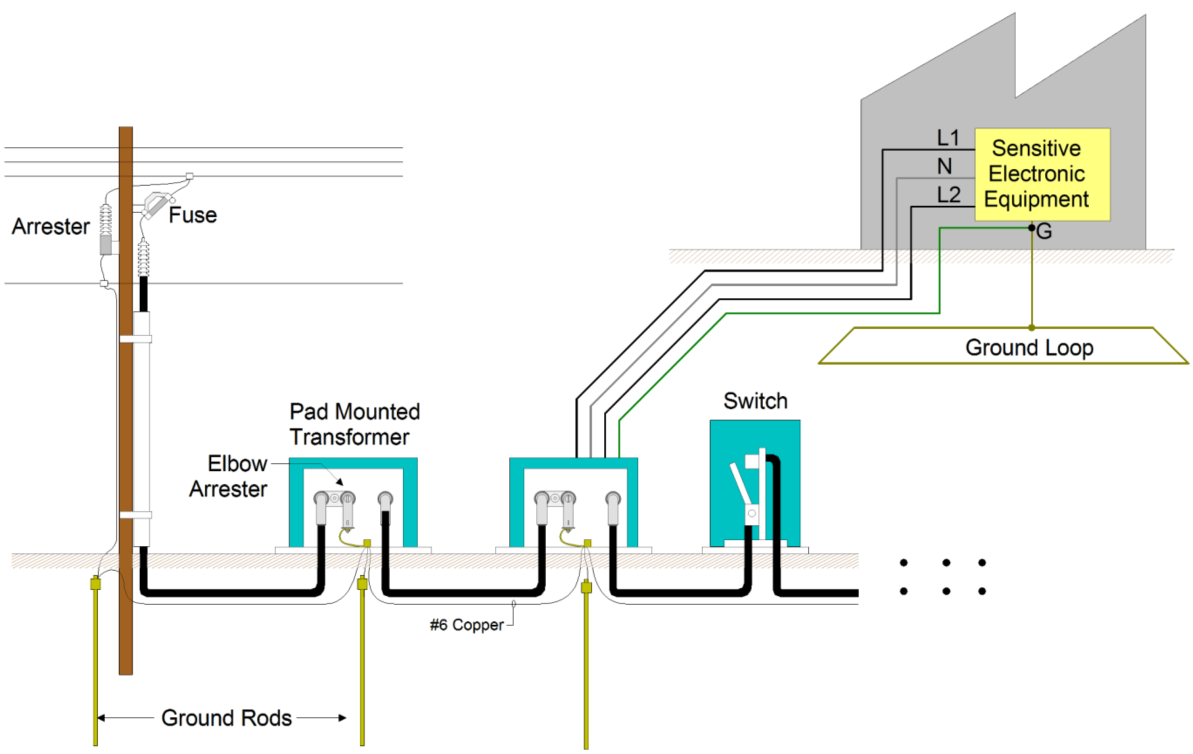

Figure 3.2: Typical Overhead and Underground Distribution Systems

We alluded to the fact that the component models should reproduce the response of the components to specific inputs and phenomena with high fidelity. It is important to recognize that the selection of the appropriate model depends on the phenomena to be studied. For example, to study power frequency phenomena in a transformer, a simple model will suffice. If, however, high frequency phenomena are to be studied, then a totally different transformer model will be necessary. Similarly, if transformer in-rush currents are to be computed, a totally different model must be utilized, specifically one that captures the nonlinearity of the transformer core and properly represent the dependence of the magnetizing current on the magnetic flux of the transformer core. The model selection also depends on the time period of concern. For example, if the steady state of the system is to be analyzed, appropriate steady state models should be used. If, however, inrush current phenomena in transformers are to be studied, another set of models must be employed. Therefore one should realize that the phenomena under study and the time period of concern will determine the selection of the proper model. The most usual phenomena under study and time periods of concern are listed below.

Phenomena Under Study

- Power Frequency
- Line Switching
- Capacitor Bank Switching
- Transient Recovery Voltage
- Lightning

Period of Concern

- Steady State
- Short Term (seconds)
- Milliseconds
- Microseconds

In the rest of this chapter, models of the most usual power system components will be introduced with comments about their applicability to specific phenomena under study. It should be understood, that most of the phenomena that affect power quality are typically of relatively low frequency.

3.2 Transmission and Distribution Line Modeling

Transmission and distribution lines can be of many varieties: overhead three phase, single phase, underground three phase cables, underground single phase cables, etc. The distinction between transmission and distribution depends on the intended purpose of the power circuit. Specifically if the intended use is to supply customers (residential, commercial and industrial) then we refer to this line as distribution. In general distribution lines operate at medium voltage (a few kVs to about 35 kV) and most times they operate radially. Power circuits operating at higher voltages are typically classified as transmission circuits. Mathematically, the methods for modeling

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transmission and distribution circuits are identical. We present some typical transmission and distribution lines and then we address the modeling of these components.

The components of overhead transmission lines and distribution lines are illustrated in Figure 3.3. A three-phase overhead line consists of three phase conductors HA, HB, and HC, which are suspended with insulators from towers. Most designs include an overhead ground wire (OHGW (**O**ver**H**ead **G**round **W**ire) or shield wire) to provide protection against lightning. Many OHGW also include a tube with optical fibers for communications. The OHGW is typically connected to the neutral of the system and may be grounded at each tower. The tower grounding system may consist of counterpoise (illustrated in Figure 3.3), rings, ground rods, etc. A typical overhead transmission line terminates to two substations. The OHGW is typically connected to the grounding system of the substations. Figure 3.3 illustrates the termination of the OHGW to the substation ground mat. A three-phase overhead distribution line is also illustrated in Figure 3.3. It consists of three phase conductors, indicated as LA, LB, and LC, and a multiply grounded neutral conductor. The neutral conductor is typically bonded to the substation ground mat and to the grounds of the distribution poles.

Overhead power lines are suspended on towers or poles. The design of transmission towers depends on the operating voltage of the line and other mechanical strength considerations. Three example tower/pole designs are illustrated in Figures 3.4, 3.5, and 3.6 for 230-kV, 115-kV, and 12-kV lines, respectively. Note that the 12-kV line, which is typically used in distribution circuits, does not have an OHGW. Instead, it has a fourth conductor, the neutral, which is suspended below the phase conductors. While electrically the OHGW and neutral are similar, the naming difference reflects the fact that the OHGW is not intended to carry electric current under normal operating conditions while the neutral is intended to carry the return current under normal operating conditions. The size of the neutral conductor is comparable to that of the phase conductors and it is intended to carry potentially the full load current. The reason for this practice is the fact that distribution circuits may supply single phase loads connected between a phase and the neutral conductor. This practice generates unbalanced conditions and the neutral conductor may carry a substantial electrical current.

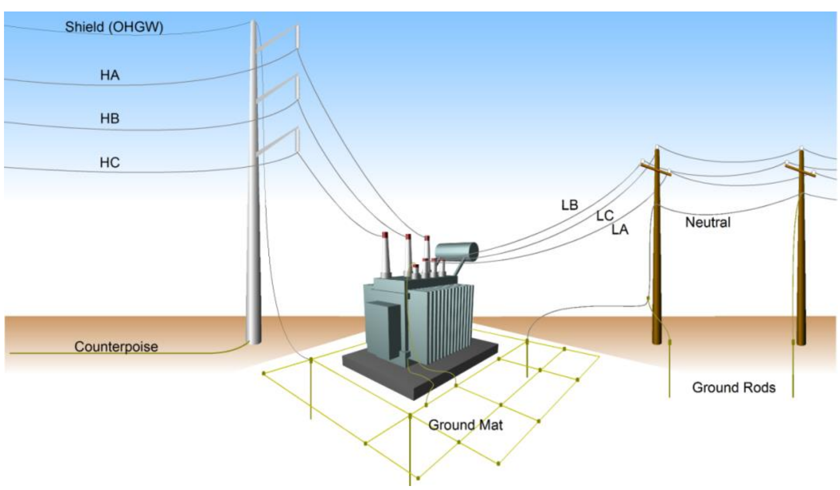

Figure 3.3: Overhead Power Lines: a Transmission and a Distribution Line Connected via a Power Transformer

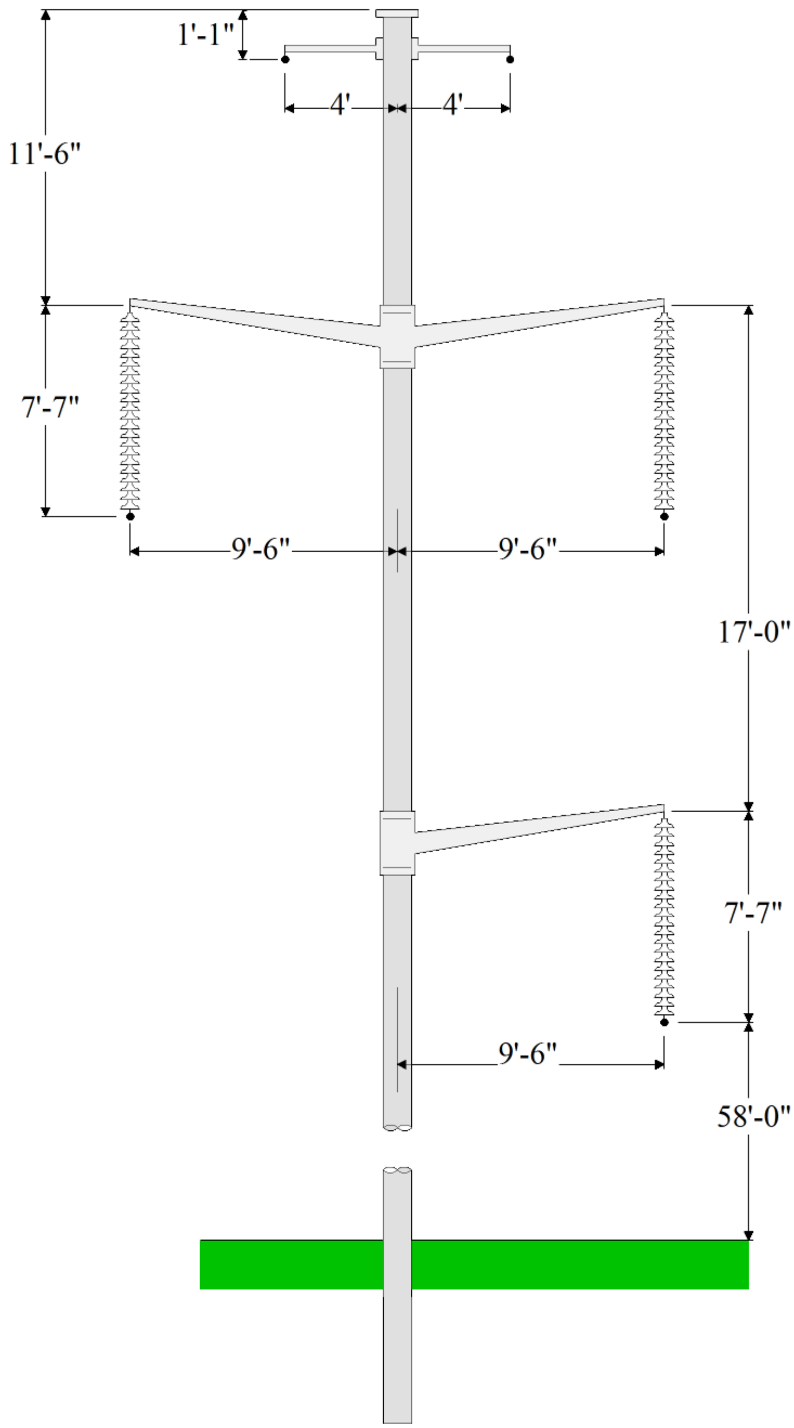

Figure 3.4: Design of a 230-kV H-frame Transmission Tower
(Courtesy of Georgia Power Co.)

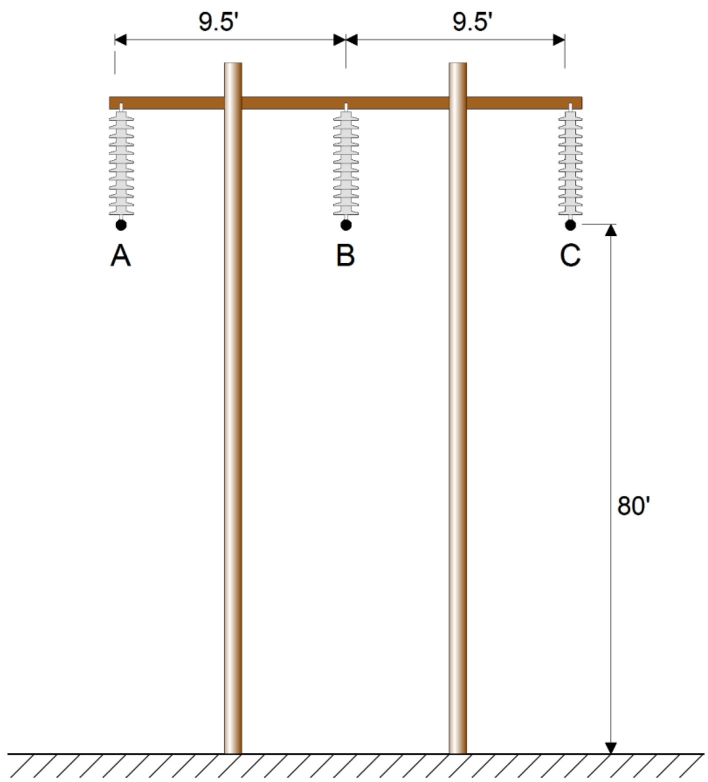

Figure 3.5: Design of a 115-kV H-Frame Transmission Tower

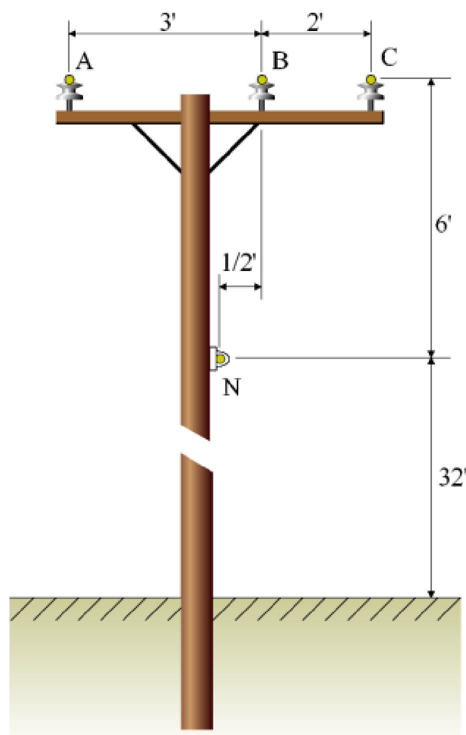

Figure 3.6: Design of a 12-kV Single-Pole Distribution

Recent advances in technology have made DC transmission an economically attractive alternative over long distances. A typical DC transmission line is illustrated in Figure 3.7. It consists of two bundle conductors, the positive and negative poles, and an overhead ground conductor.

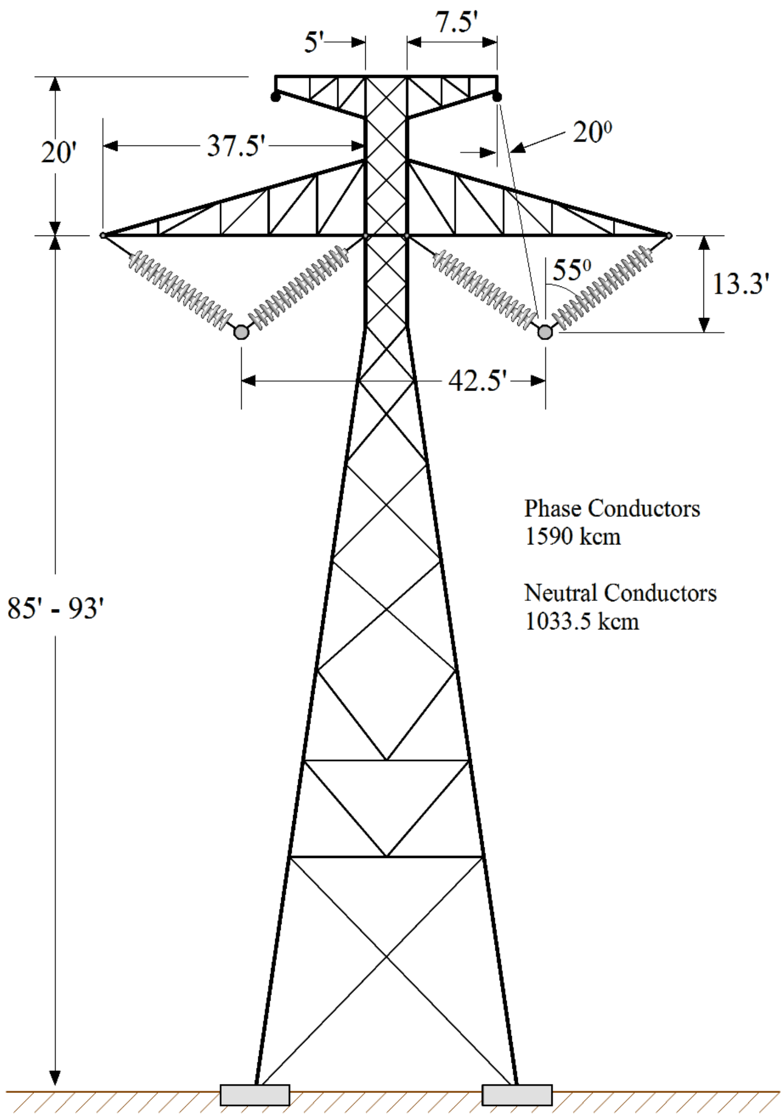

Figure 3.7: Design of a ±400-kV HVDC Tower
(Courtesy of the Electric Power Research Institute)

Power lines can be also constructed from power cables. Cables may be three phase, or single phase cables connected in a three phase arrangement. A typical three phase construction with three single phase power cables is illustrated in Figure 3.8a and a typical three phase power cable construction is illustrated in Figure 3.8b.

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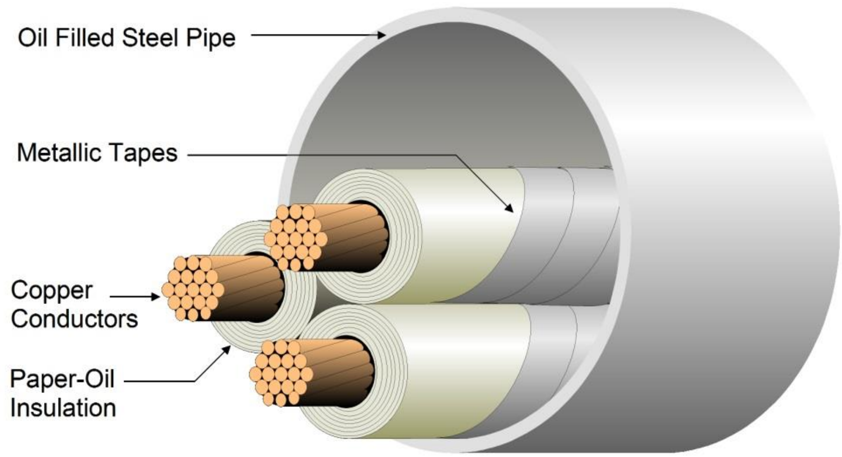
(a)



(b)

Figure 3.8: Typical Power Cables: (a) 3-Single Phase Solid Dielectric, (b) Three Phase Oil Filled

A distribution system comprises power lines and voltage-step-down equipment for electric service at industrial, commercial, and residential sites. A distribution system may comprise three-phase transmission lines, with typical operating voltages of 12 to 35 kV line to line, and three-phase, two phase, or single phase tapped lines. The construction of these lines may be overhead or underground. These possibilities are illustrated in Figure 3.2. Figure 3.2 suggests that distribution systems may operate (and in fact they do operate) under unbalanced conditions. Some of this imbalance may transmit to the transmission system. This means that distribution systems present some unique analysis problems. In addition, recent advances in end-use equipment technology have resulted in electric loads that may be interacting with the system dynamically. For example, solid-state motor controllers, rectifiers, and so on, inject harmonics into the distribution system. Analysis and understanding of these phenomena require that the distribution system be modeled and understood not only for the power frequency (60-Hz in the United States, 50 Hz in Europe) but also for other frequencies, such as the harmonics of 60 Hz.

For several technical and safety reasons, electric power installations must be grounded. Grounding of power systems is achieved by embedding metallic structures (conductors) into earth and electrically connecting these conductors to the neutral of the power system. In this way a low impedance is provided between the power system neutral and the vast conducting soil,

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which guarantees that the voltage of the neutral, with respect to earth, will be low under all conditions. Grounding is necessary for several reasons: (a) to assure correct operation of electrical devices, (b) to provide safety during normal or fault conditions, (c) to stabilize the voltage during transient conditions, and (d) to dissipate lightning strokes. An example of the physical construction of a substation with the underlying grounding system is illustrated in Figure 3.9.

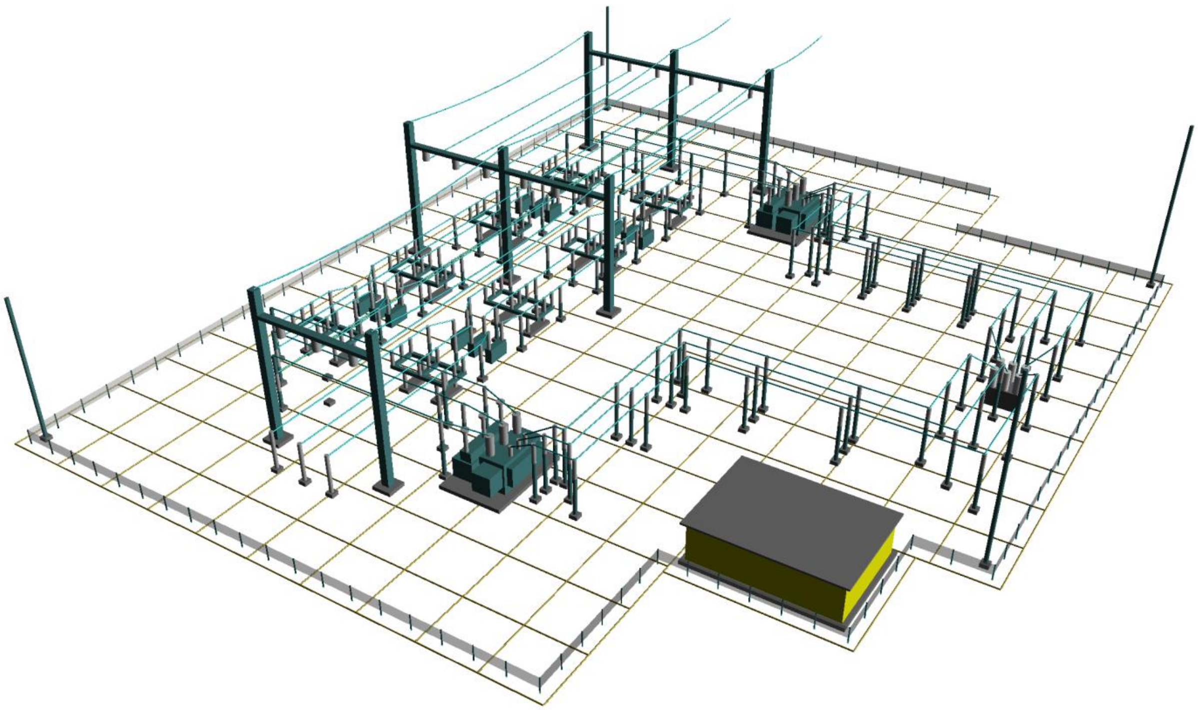

Figure 3.9: Example of the Physical Arrangement of a Substation Illustrating the Grounding, Fences and Electrical Equipment

The described physical structures are typically modeled with proper mathematical models. The presentation of line modeling will be done in several steps. First, we shall examine the per unit length parameters of a power line. These parameters are: resistance, inductance, and capacitance.

Next, analysis procedures will be introduced by which equivalent circuits of power lines will be developed. Depending on the objectives of the analysis the mathematical models may be different for the same physical structure. As an example for analysis of a power lines under steady state 60 Hz sinusoidal operation, a π-equivalent circuit completely captures the behavior of the line. However for the same line, this equivalent circuit is inadequate to describe transients on the line. In general, the following models of transmission and distribution lines and relative applications may be encounter:

1. Three-phase power lines can be approximated in terms of their sequence equivalent circuits (positive, negative and zero sequence). These models represent an approximation of the actual behavior of a line. They are extensively used for power flow studies, short-circuit analysis, and stability studies.

2. Power lines can be also modeled with explicit representation of transmission tower, neutral wires or ground wires, grounding systems and substation grounding systems. These models are applicable for ground potential rise computations, safety analysis and for design of grounding systems [???].
3. Distributed parameter models of power lines can be also developed. These models are applicable for fast electrical transient analysis, (such as switching transients, lightning transients) and the design of overvoltage protection. These models will not be considered in this book.

In this section, the basic equations of a transmission line model are presented for low frequencies. We focus on the derivation of the resistance, inductance and capacitance of the line and subsequent extraction of appropriate equivalent circuits.

3.2.1 Resistance

The resistance of power conductors is dependent upon the frequency of the electric current. For example the DC resistance (r_{dc}, f=0 Hertz) can be directly computed from the conductor material resistivity:

$$r_{dc} = \rho \frac{1}{A} \, ohms/meter$$

where ρ is the resistivity of the conductor material and A is the cross section of the conductor.

The computation of the AC resistance, r_{ac}, of a power conductor can be quite complicated, depending on the geometry (cross section) of the conductor. For cylindrical conductors, the AC resistance of the conductor is given in terms of Bessel functions:

$$r_{ac} = r_{dc} \frac{ka}{2} \frac{M_0(ka)}{M_1(ka)} \sin\left(\theta_1(ka) - \theta_0(ka) - \frac{\pi}{4} \right) \; ohms/meter$$

where: $k = \sqrt{\omega\mu\sigma}$, $\omega = 2\pi f$, a is the radius of the conductor

Note ka is a pure number (dimensionless)

$M_0(ka), \;\; \theta_0(ka)$: are the magnitude and phase respectively of the modified Bessel function, order zero and argument ka.

$M_1(ka), \;\; \theta_1(ka)$: are the magnitude and phase respectively of the modified Bessel function, order one, argument ka.

Tabulation of these functions can be found in the references. For convenience, the values of these functions for the argument value up to 10 are provided in Table 3.1. Derivation of above equations for the ac resistance of cylindrical conductors can be found in [???].

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For other conductor cross section geometries, the reader is encouraged to consult the references.

Table 3.1: Modulus and Phase of Modified Bessel Functions

z	$M_0(z)$	$\theta_0(z)$	$M_1(z)$	$\theta_1(z)$	z	$M_0(z)$	$\theta_0(z)$	$M_1(z)$	$\theta_1(z)$
0.000	1.0000	0.00	0.0000	135.00	1.300	1.0438	23.75	0.6548	147.07
0.025	1.0000	0.01	0.0125	135.00	1.350	1.0508	25.54	0.6808	148.02
0.050	1.0000	0.04	0.0250	135.02	1.400	1.0586	27.37	0.7070	148.99
0.075	1.0000	0.08	0.0375	135.04	1.450	1.0672	29.26	0.7333	150.00
0.100	1.0000	0.14	0.0500	135.07	1.500	1.0767	31.19	0.7598	151.04
0.125	1.0000	0.22	0.0625	135.11	1.550	1.0871	33.16	0.7866	152.12
0.150	1.0000	0.32	0.0750	135.16	1.600	1.0984	35.17	0.8136	153.23
0.175	1.0000	0.44	0.0875	135.22	1.650	1.1108	37.22	0.8408	154.38
0.200	1.0000	0.57	0.1000	135.29	1.700	1.1242	39.30	0.8684	155.55
0.225	1.0000	0.73	0.1125	135.36	1.750	1.1387	41.41	0.8962	156.76
0.250	1.0001	0.90	0.1250	135.45	1.800	1.1544	43.54	0.9244	158.00
0.275	1.0001	1.08	0.1375	135.54	1.850	1.1712	45.70	0.9530	159.27
0.300	1.0001	1.29	0.1500	135.64	1.900	1.1892	47.88	0.9819	160.57
0.325	1.0002	1.51	0.1625	135.76	1.950	1.2085	50.08	1.0113	161.90
0.350	1.0002	1.75	0.1750	135.88	2.000	1.2290	52.29	1.0412	163.27
0.375	1.0003	2.01	0.1875	136.01	2.050	1.2509	54.51	1.0715	164.66
0.400	1.0004	2.29	0.2000	136.15	2.100	1.2741	56.74	1.1024	166.08
0.425	1.0005	2.59	0.2125	136.29	2.150	1.2986	58.98	1.1339	167.53
0.450	1.0006	2.90	0.2250	136.45	2.200	1.3246	61.22	1.1659	169.00
0.475	1.0008	3.23	0.2375	136.62	2.250	1.3520	63.46	1.1987	170.50
0.500	1.0010	3.58	0.2500	136.79	2.300	1.3808	65.71	1.2321	172.03
0.525	1.0012	3.95	0.2626	136.97	2.350	1.4111	67.95	1.2663	173.58
0.550	1.0014	4.33	0.2751	137.17	2.400	1.4429	70.19	1.3012	175.16
0.575	1.0017	4.73	0.2876	137.37	2.500	1.5111	74.65	1.3736	178.39
0.600	1.0020	5.15	0.3001	137.58	2.600	1.5855	79.09	1.4498	181.70
0.625	1.0024	5.59	0.3126	137.80	2.700	1.6665	83.50	1.5300	185.10
0.650	1.0028	6.04	0.3252	138.03	2.800	1.7541	87.87	1.6148	188.57
0.675	1.0032	6.52	0.3377	138.26	2.900	1.8486	92.21	1.7046	192.11
0.700	1.0037	7.01	0.3502	138.51	3.000	1.9502	96.52	1.7999	195.71
0.725	1.0043	7.51	0.3628	138.76	3.100	2.0593	100.79	1.9011	199.37
0.750	1.0049	8.04	0.3753	139.03	3.200	2.1760	105.03	2.0088	203.08
0.775	1.0056	8.58	0.3879	139.30	3.300	2.3009	109.25	2.1236	206.83
0.800	1.0064	9.14	0.4004	139.58	3.400	2.4342	113.43	2.2458	210.62
0.825	1.0072	9.72	0.4130	139.87	3.500	2.5764	117.60	2.3763	214.44
0.850	1.0081	10.31	0.4256	140.17	3.600	2.7280	121.75	2.5155	218.30
0.875	1.0091	10.92	0.4382	140.48	3.700	2.8894	125.87	2.6640	222.17
0.900	1.0102	11.55	0.4508	140.80	3.800	3.0613	129.99	2.8227	226.07
0.925	1.0114	12.19	0.4634	141.12	3.900	3.2443	134.10	2.9920	229.98
0.950	1.0127	12.86	0.4760	141.46	4.000	3.4391	138.19	3.1729	233.90
0.975	1.0140	13.53	0.4886	141.80	4.500	4.6179	158.59	4.2783	253.67
1.000	1.0155	14.23	0.5013	142.16	5.000	6.2312	178.93	5.8091	273.55
1.025	1.0171	14.94	0.5140	142.52	5.500	8.4473	199.28	7.9253	293.48
1.050	1.0188	15.66	0.5267	142.89	6.000	11.5008	219.62	10.8502	313.45
1.075	1.0207	16.40	0.5394	143.27	6.500	15.7170	239.96	14.8961	333.46
1.100	1.0227	17.16	0.5521	143.66	7.000	21.5479	260.29	20.5003	353.51
1.125	1.0248	17.93	0.5648	144.05	7.500	29.6223	280.61	28.2737	373.59
1.150	1.0270	18.72	0.5776	144.46	8.000	40.8176	300.92	39.0697	393.69
1.175	1.0294	19.52	0.5904	144.87	8.500	56.3586	321.22	54.0807	413.82
1.200	1.0320	20.34	0.6032	145.29	9.000	77.9565	341.52	74.9740	433.96
1.225	1.0347	21.17	0.6161	145.73	9.500	108.0039	361.81	104.0822	454.11
1.250	1.0376	22.02	0.6290	146.17	10.000	149.8476	382.10	144.6705	474.28

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3.2.2 Inductance

We examine the modeling methods for computing the inductance of transmission circuits. First the fundamentals are presented, followed by more practical analysis methods.

3.2.2.1 Basic Magnetic Field Equation around a Conductor

Conceptually, the phenomena to be studied can be explained through the simple two-conductor line illustrated in Figure 3.10. Assume that electric current i(t), which is time dependent, flows through one conductor and returns through the other conductor. The current flow generates a magnetic field that is time dependent, i.e. it follows the time variation of the electric current. Consider an infinitesimal length dx of conductor. Let dλ(t) be the magnetic flux linking the electric current i(t) flowing in the infinitesimal length dx of the conductor. By definition, the inductance of the length dx of the conductor is dL, where

$$dL = \frac{d\lambda(t)}{i(t)} \tag{3.1}$$

Since the magnetic flux linkage is time varying, a voltage dv(t) will be induced along length dx of the conductor:

$$dv(t) = \frac{d\lambda(t)}{dt} = dL\frac{di(t)}{dt}$$

Now assume that the inductance of the conductor is L henries per meter; then

$$dL = L\ dx$$

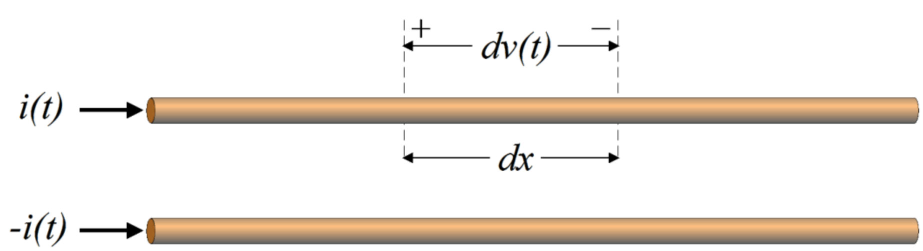

Figure 3.10: A Simple Two Conductor Line

Upon substitution in the equations above and subsequent solution for L, we have

$$L = \frac{\dfrac{dv(t)}{dx}}{\dfrac{di(t)}{dt}} \quad \text{H/m (Henries/meter)} \tag{3.2}$$

Equation (3.1) or (3.2) defines the inductance of a conductor. Specifically, Eq. (3.1) states that the inductance equals the magnetic flux linkage divided by the electric current. Alternatively, Equation (3.2) states that the inductance equals the induced voltage per unit length divided by the time derivative of the electric current.

A transmission line is a complicated structure, comprising two or more conductors. Our objective in this chapter is to characterize each conductor with its inductance and also any pair of conductors with a mutual inductance.

We introduce the basic concepts by considering the magnetic field of an infinity long conductor of circular cross section. For simplicity, assume that the conductor material is nonmagnetic. In other words, the permeability of the conductor material is μ_0. A cross section of the conductor is shown in Figure 3.11a. The radius of the conductor is a. Further assume that the conductor carries an electric current i(t), which is uniformly distributed in the cross section of the conductor (i.e. constant current density). Under these assumptions, it is relatively easy to compute the magnetic field of the configuration and subsequently the inductance of the line.

Because of the existing cylindrical symmetry, the magnetic field intensity $\underline{H}$ at a point A, illustrated in Figure 3.11a, will be perpendicular to the radial direction and the magnitude will be constant on the circular contour with center O and radius r. In other words, the magnitude of the magnetic field intensity, H, is a function of the radius r only [i.e. H(r)]. H(r) is computed with a direct application of Ampere's law on the described configuration. There are two cases.

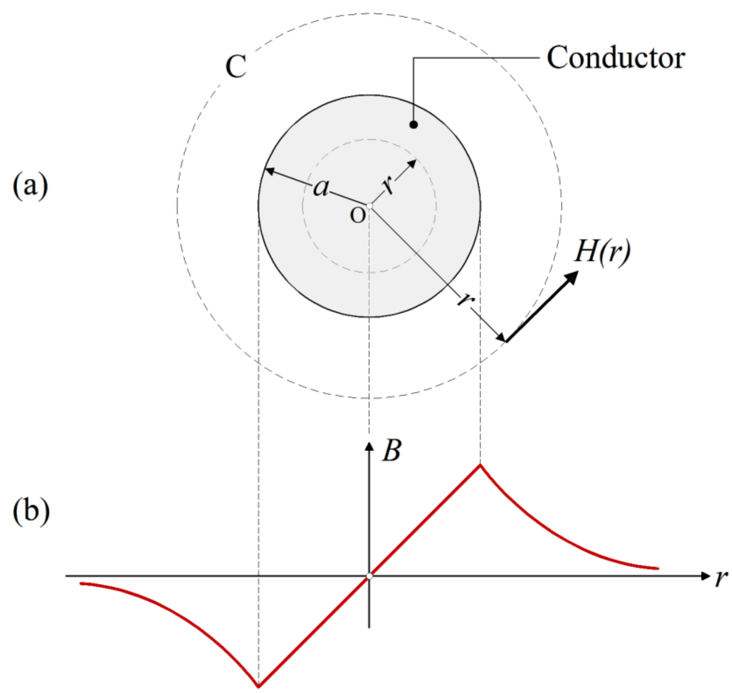

Figure 3.11: Infinitely Long Circular Conductor
[(a) Cross Section, (b) Magnetic Flux Density Along a Radial Direction]

Case a. The point A is located outside the conductor:

$$r \geq a$$

Application of Ampere's law yields

$$i(t) = \int_C \underline{H}(r) \cdot d\underline{\ell} = 2\pi r H(r)$$

Upon solution for H(r), we obtain:

$$H(r) = \frac{i(t)}{2\pi r}, \quad r > a \tag{3.3}$$

The magnetic flux density is given by

$$B(r) = \mu_0 H(r) = \mu_0 \frac{i(t)}{2\pi r}, \quad r > a \tag{3.4}$$

Case b. The point A is located inside the conductor:

$$r \leq a$$

Application of Ampere's law yields:

The electric current inside C_2: $\quad i_{C_2} = \int_{C_2} \underline{H}(r) \cdot d\underline{\ell} = 2\pi r H(r)$

In general the computation of the current inside the curve C_2 may be quite complicated. For simplicity and for low frequencies, we introduce the simplifying assumption that the electric current density is constant inside the conductor In this case:

The electric current inside C_2: $\quad i_{C_2} = \dfrac{\pi r^2}{\pi a^2} i(t) = \left(\dfrac{r}{a}\right)^2 i(t), \quad r \leq a$

Substitution and subsequent solution for H(r) yields

$$H(r) = \frac{1}{2\pi a}\left(\frac{r}{a}\right) i(t), \quad r \leq a \tag{3.5}$$

and

$$B(r) = \mu_0 H(r) = \frac{\mu_0}{2\pi a}\left(\frac{r}{a}\right) i(t), \quad r \leq a \tag{3.6}$$

The results are summarized in Figure 3.11b, where the magnetic flux density B(r) is plotted as a function of r along a radial direction.

From the magnetic flux density B, the magnetic flux Φ crossing any surface S is computed from the integral

$$\Phi = \int_S B \bullet ds$$

If the surface S crosses the conductor and since the electric current is distributed inside the conductor, the magnetic flux will link variable portions of the electric current. In this case the use of the concept of magnetic flux linkage is expedient. The magnetic flux linkage is defined by

$$\lambda = \int_S wB \bullet ds$$

where w is the portion of electric current linked with the infinitesimal magnetic flux $B \bullet ds$.

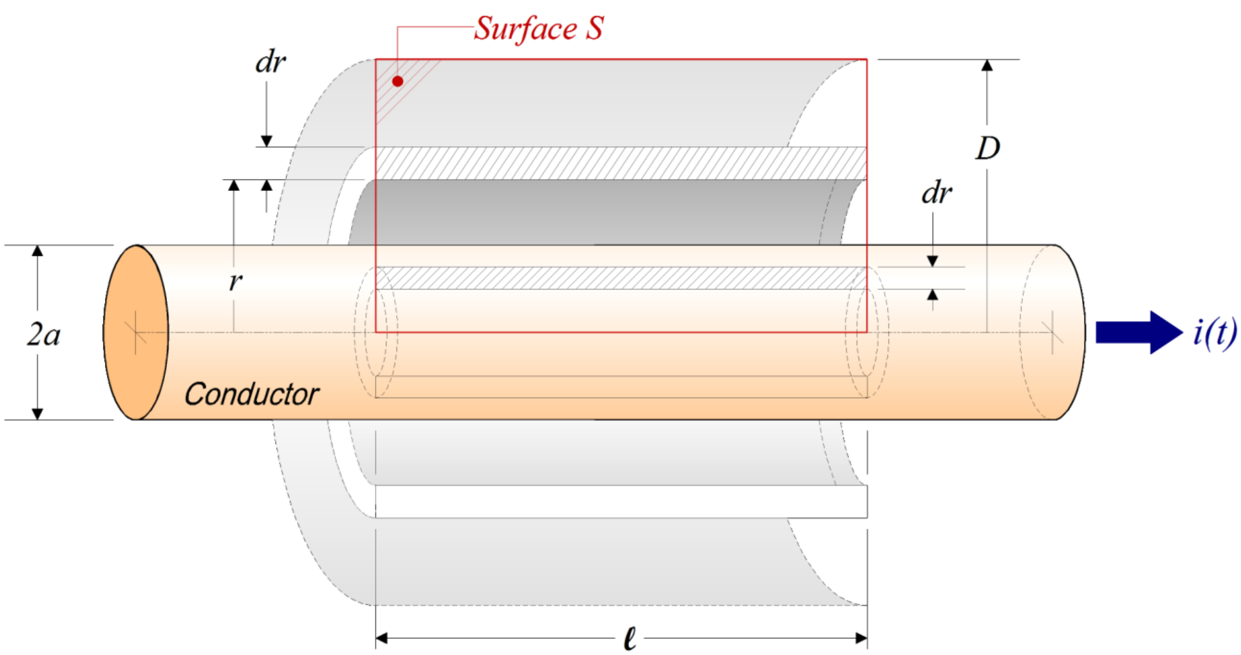

Figure 3.12: Geometry of surface S

Given the magnetic flux linkage though a surface S, the induced voltage v(t) along the perimeter of the surface is computed by

$$v(t) = \frac{d\lambda(t)}{dt}$$

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As an example, consider a rectangular surface S, of dimensions ℓ and D, located on a plane passing through the axis of the conductor. The surface S is defined in Figure 3.12. Consider the two illustrated infinitesimal strips of the area ℓdr located on the surface S and parallel to the axis of the conductor. One infinitesimal strip is located inside the conductor at a distance $r_1 \leq a$ from the axis. The magnetic flux through the infinitesimal strip $dS_1 = \ell dr \; at \; r = r_1 \leq a$ (inside the conductor), links $\frac{\pi r^2}{\pi a^2}$ portion of the electric current. Thus the magnetic flux linkage $d\lambda_{\text{int}}(t)$ is

$$d\lambda_{\text{int}}(t) = \frac{\pi r^2}{\pi a^2} B(r)\ell = \frac{\mu_0 r^3 i(t)}{2\pi a^4} \ell dr$$

The magnetic flux linkage of the second infinitesimal strip $dS_2 = \ell dr \; at \; r = r_2 \geq a$ (outside the conductor), links the entire electric current through the conductor. The magnetic flux linkage of this infinitesimal strip $d\lambda_{ext}(t)$ is

$$d\lambda_{ext}(t) = \frac{\mu_0 i(t)}{2\pi r} \ell dr$$

The total magnetic flux linkage through the surface S is

$$\lambda(t) = \int_{r=0}^{a} \frac{\mu_0 r^3 i(t)}{2\pi a^4} \ell dr + \int_{r=a}^{D} \frac{\mu_0 i(t)}{2\pi r} \ell dr$$

Evaluation of the integrals provides the following result:

$$\lambda(t) = \frac{\mu_0 \ell i(t)}{2\pi} \left(\frac{1}{4} + \ln\left(\frac{D}{a} \right) \right) \qquad (3.7)$$

Equation (3.7) is usually written in the following compact form:

$$\lambda(t) = \frac{\mu_0 \ell i(t)}{2\pi} \ln\left(\frac{D}{d} \right), \quad d = ae^{-\frac{1}{4}} \qquad (3.8)$$

The quantity d is known as the geometric mean radius of the conductor. The physical meaning of the geometric mean radius is that a thin hollow conductor of radius equal to the geometric mean radius and carrying the same electric current i(t), produces the same magnetic flux linkage as the conductor under consideration. This interpretation will be illustrated by the following example.

Example E3.1: An infinitely long hollow conductor of average radius d and infinitesimal thickness carries as electric current i(t). The conductor is illustrated in Figure E3.1a. For clarity, it is shown with finite thickness.

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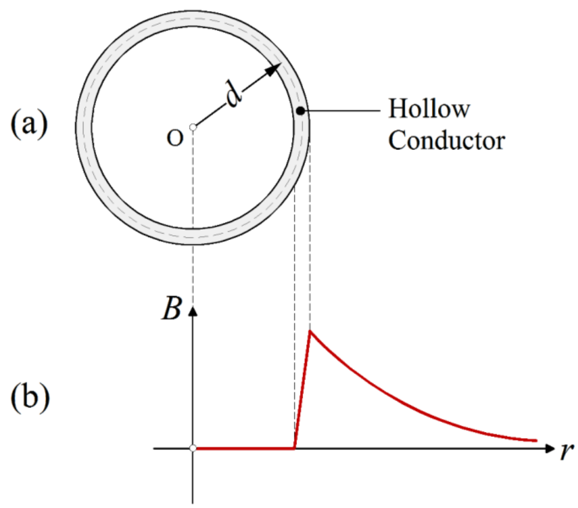

Figure E3.1 Magnetic Field around a Hollow Conductor Carrying Electric Current

Show that the magnetic flux linking a rectangular surface of dimensions ℓ and D, with one ℓ-long side located on the axis of the conductor, is

$$\lambda(t) = \frac{\mu_0 \ell i(t)}{2\pi} \ln\left(\frac{D}{d}\right)$$

Solution: The magnetic field density around this configuration is illustrated in Fig. E3.1b. Specifically, the magnetic field density is

$$B(r) = \begin{bmatrix} 0, & r \le d \\ \dfrac{\mu_0 i(t)}{2\pi r}, & r \ge d \end{bmatrix}$$

The magnetic flux linkage is

$$\lambda(t) = \int_{r=d}^{D} \frac{\mu_0 i(t)}{2\pi r} \ell dr = \frac{\mu_0 \ell i(t)}{2\pi} \ln\left(\frac{D}{d}\right)$$

This completes the proof.

The induced voltage across the conductor due to the magnetic flux is readily computed from

$$v(t) = \frac{d\lambda(t)}{dt} = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{D}{d}\right) \frac{di(t)}{dt}$$

By definition, the inductance of the conductor is

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$$L_t = \frac{\lambda(t)}{i(t)} = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{D}{d}\right)$$

On a per unit length basis, the inductance is

$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{D}{d}\right) \tag{3.10}$$

One should observe that the inductance of the conductor is dependent on the width D of the selected surface S. Since the width D can be selected arbitrarily, the result above does not have any physical meaning. This peculiarity occurs because the path of return of the electric current i(t) has been neglected. It is apparent that in order to compute the inductance of the conductor in a unique and meaningful way, it is necessary to consider the entire circuit which includes the path of return of the electric current. In any practical situation, all conductors or objects carrying electric current will be located in a finite area. In this case, as we shall see in subsequent sections, the inductance of the conductors can be uniquely defined. Despite the lack of realism of the configuration being considered, the results obtained are fundamental for the computation of the inductances of realistic transmission line configurations, as we shall see in subsequent sections.

In summary we have derived expressions for the magnetic field density and magnetic flux linkage of a current carrying conductor. We will use these results for the analysis of practical transmission lines.

3.2.2.2 Inductive Equations of a Multi-Conductor Line

In general a line configuration involves multiple conductors. Each conductor carries a certain electric current. Because of physical considerations (conservation of charge) the sum of the electric currents must be equal to zero. Such an arrangement is shown in Figure 3.13. The current of each conductor will establish a magnetic field around it which will link all other conductors. The net result will be an induced voltage on each conductor. Considering conductor j, the induced voltage will be along the conductor as it is shown in Figure 3.13. For computing this voltage one must determine the magnetic flux linkage per unit length of the conductor.

Consider a rectangular frame with one side of the frame located on the axis of conductor j. The frame extends to a distance x from the axis of the conductor and its length is l. The flux linkage through this frame with respect to the current through conductor j, i.e. the flux linkage of conductor j will be

$$\lambda_{jx}(t) = \lambda_{jjx}(t) + \sum_k \lambda_{jkx}(t)$$

where $\lambda_{jkx}(t)$ is the contribution of conductor k to the flux linkage of conductor j.

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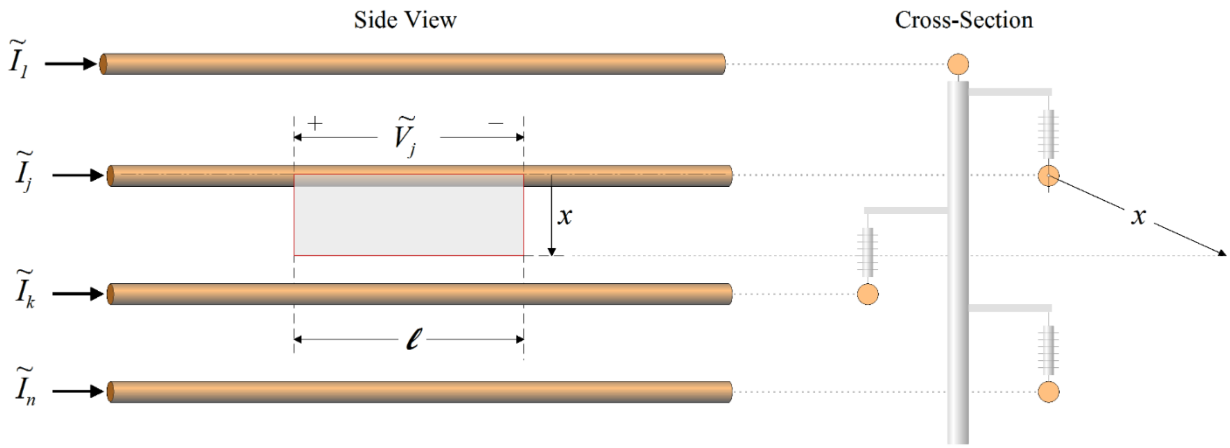

Figure 3.13: Illustration of Induced Voltage

To compute this term consider Figure 3.14, which illustrates the cross section of the system of conductors (only conductors j and k are shown) and the frame jx. We would like to determine the flux linkage through the frame jx defined with the axis of conductor j and a line parallel to conductor j passing through point x. Note that the contribution to the magnetic flux linkage from the current of conductor j is:

$$\lambda_{jjx}(t) = \frac{\mu_0 \ell i_j(t)}{2\pi} \ln\left(\frac{D_{jx}}{d_j}\right)$$

Also note that the contribution to the magnetic flux linkage of conductor j from the electric current of conductor k is the magnetic flux linkage through the surface defined with the line d_{jk}. This magnetic flux equals the flux linkage through the line mx which is given by

$$\lambda_{jkx}(t) = \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{D_{kx}}{d_{jk}}\right)$$

Note that the distance d_{km} is the same as the distance d_{jk}. The total magnetic flux linkage through the frame jx can be formed from the contribution to the flux from all conductors, i.e.:

$$\lambda_{jx}(t) = \frac{\mu_0 \ell i_j(t)}{2\pi} \ln\left(\frac{D_{jx}}{d_j}\right) + \sum_k \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{D_{kx}}{d_{jk}}\right)$$

The above equation can be written in compact form as follows:

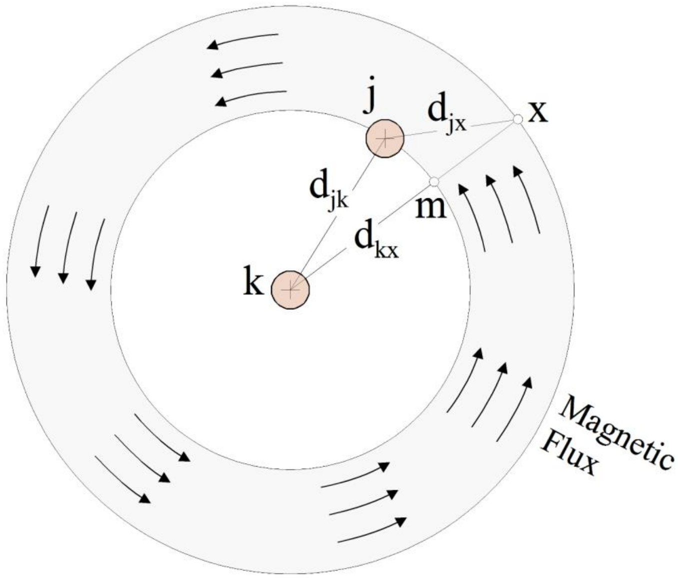

Figure 3.14: Illustration of Magnetic Flux Through Plane d_{jx} due to Electric Current i_k (t)

$$\lambda_{jx}(t) = \sum_{k=1}^{n} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{D_{kx}}{d_{jk}}\right)$$

where n is the number of conductors, d_{jk} is the distance between conductors j, k if $j \neq k$, and d_{jj} is the geometric mean radius of conductor j.

It is easy to prove that under the observation that $\sum_{k=1}^{n} i_k(t) = 0$ and as the point x goes to infinity:

$$\lambda_{jx}(t) \rightarrow \lambda_j(t) = \sum_{k=1}^{n} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{1}{d_{jk}}\right)$$

Proof: Since the sum of all currents equals zero, then the current of the last conductor n can be written as the negative sum of all other currents:

$$i_n(t) = -\sum_{k=1}^{n-1} i_k(t)$$

Upon substitution in the expression for the magnetic flux linkage:

$$\lambda_{jx}(t) = \sum_{k=1}^{n-1} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{d_{kx}}{d_{jk}}\right) - \sum_{k=1}^{n-1} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{d_{nx}}{d_{jn}}\right)$$

The above expression can be rewritten in the following form:

$$\lambda_{jx}(t) = \sum_{k=1}^{n-1} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{1}{d_{jk}}\right) - \sum_{k=1}^{n-1} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{1}{d_{jn}}\right) + \sum_{k=1}^{n-1} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{d_{kx}}{d_{nx}}\right)$$

Note that the last sum will vanish as the point x goes to infinity because each term will become zero (logarithm of 1.0). The second sum can be expressed in terms of the current in conductor n. Thus:

$$\lambda_j(t) = \sum_{k=1}^{n} \frac{\mu_0 \ell i_k(t)}{2\pi} \ln\left(\frac{1}{d_{jk}}\right)$$

This concludes the proof.

The induced voltage along the conductor is computed as the time derivative of the magnetic flux linkage of the conductor.

$$v_j(t) = \frac{d\lambda_j(t)}{dt} = \sum_{k=1}^{n} \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{1}{d_{jk}}\right) \frac{di_k(t)}{dt}$$

Assuming sinusoidal steady state conditions,

$$v_j(t) = \mathrm{Re}\left[\sqrt{2}\tilde{V}_j e^{j\omega t}\right]$$

$$i_k(t) = \mathrm{Re}\left[\sqrt{2}\tilde{I}_k e^{j\omega t}\right]$$

Upon substitution with manipulations

$$\tilde{V}_j = \sum_{k=1}^{n} \frac{j\omega\mu_0 \ell}{2\pi} \ln\left(\frac{1}{d_{jk}}\right) \tilde{I}_k = \sum_{k=1}^{n} x_{jk} \tilde{I}_k$$

where

$$x_{jk} = \frac{j\omega\mu_0 \ell}{2\pi} \ln\left(\frac{1}{d_{jk}}\right)$$

The previous results can be directly used to determine the induced per unit length of any line.

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3.2.2.3 Inductive Equations of a Multi-Conductor Line Above Earth

Overhead or underground power transmission lines are characterized by the fact that earth is one of the paths for the flow of electric current. Electric current can flow into the soil through the grounding system (substation ground mat, pole grounds, etc.) and vice versa can flow into the shield/neutral wires from the soil. During normal operating conditions, some electric current flows in the conductive earth soil. This current it is generally generated by a combination of inductive, conductive and capacitive phenomena. In general, the magnitude of the earth current during normal operating conditions is comparatively low. During abnormal operating conditions (faults), a substantial amount of electric current may flow through earth. In any case, the earth current induces a voltage along the transmission line, thus affecting the performance of the power line. As a matter of fact, most three-phase overhead transmission circuits are designed in such a way that during ground faults the majority of the fault current flows through the earth.

The distribution of the current in the earth follows a complex, non-uniform pattern. As a result, the computation of the inductive reactance of the earth path and the mutual inductance between the earth path and overhead conductors is very complex. In this section we present the expressions for the series impedances, derived by Carson [???] and Rudenberg [???]. The results have been converted into the metric system of units (or English system) and adapted to the two-conductor system above earth as it illustrated in Figure 3.15. Specifically, consider the simplest configuration of two overhead conductors, j and k respectively, parallel to the surface of the earth and carrying electric currents $\tilde{I}_j$ *and* $\tilde{I}_k$, respectively. The configuration is illustrated in Figure 3.15a and 3.15b. Assume that there are not any other conductors in the vicinity. Then the current through the soil path, i.e. the earth current $\tilde{I}_e$, is $\tilde{I}_e = -\tilde{I}_j - \tilde{I}_k$. Carson [???] has given a solution to this problem in terms of a complex infinite series. A converted version of Carson's result (converted into the metric system of units) is provided below (only the first few terms of the infinite series are retained). Specifically, the induced voltage along the conductor a is:

$$\tilde{V}_a = \left[r_a + j\frac{\omega\mu}{2\pi}\ln\frac{D_{aa'}}{d} + \frac{\omega\mu}{\pi}(P_{aa} + jQ_{aa}) \right]\tilde{I}_a + \left[j\frac{\omega\mu}{2\pi}\ln\frac{D_{ab'}}{d_{ab}} + \frac{\omega\mu}{\pi}(P_{ab} + jQ_{ab}) \right]\tilde{I}_b$$

Where:

r_a is the conductor AC resistance at frequency ω computed as follows:

$$r_a = r_{dc}\frac{ka}{2}\frac{M_0(ka)}{M_1(ka)}\sin\left(\theta_1(ka) - \theta_0(ka) - \frac{\pi}{4}\right) ohms/meter$$

$k = \sqrt{\omega\mu\sigma}$, σ is the conductor conductivity, and ω is angular frequency of the electric current.

d is the geometric mean radius of the overhead conductor ***a***, which is calculated in terms of the conductor actual radius a as follows:

$$d = ae^{-\frac{\xi}{4}}$$

where :
$$\xi = \frac{4}{ka}\frac{M_0(ka)}{M_1(ka)}\sin\left(\theta_0(ka) - \theta_1(ka) + \frac{3\pi}{4}\right)$$

$\tilde{I}_a$ is the current through the overhead conductor ***a***, and $\tilde{I}_b$ is the current through the overhead conductor ***b***. The terms P_{aa}, Q_{aa}, P_{ab}, Q_{ab} are computed in terms of infinite series, the first few terms of which are given below:

$$P_{aa} = \frac{\pi}{8} - \frac{x}{3\sqrt{2}} + \frac{x^2}{16}(0.6728 + \ln\frac{2}{x}) + \frac{x^3}{45\sqrt{2}} - \frac{\pi x^4}{1536} + ...$$
$$Q_{aa} = -0.0386 + \frac{1}{2}\ln\frac{2}{x} + \frac{x}{3\sqrt{2}} - \frac{\pi x^2}{64} + \frac{x^3}{45\sqrt{2}} - \frac{x^4}{384} - \frac{x^4}{384}\left(\ln\frac{2}{x} + 1.0895\right) + ...$$
$$P_{ab} = \frac{\pi}{8} - \frac{y}{3\sqrt{2}}\cos\theta + \frac{y^2}{16}\cos 2\theta(0.6728 + \ln\frac{2}{y}) + \frac{y^2}{16}\theta\sin 2\theta + \frac{y^3}{45\sqrt{2}}\cos 3\theta - \frac{\pi y^4}{1536}\cos 4\theta + ...$$
$$Q_{ab} = -0.0386 + \frac{1}{2}\ln\frac{2}{y} + \frac{y}{3\sqrt{2}}\cos\theta - \frac{\pi y^2}{64}\cos 2\theta + \frac{y^3}{45\sqrt{2}}\cos 3\theta - \frac{y^4}{384}\theta\sin 4\theta - \frac{y^4}{384}\cos 4\theta(\ln\frac{2}{y} + 1.0895) + ...$$

where:

$$x = k_s D_{aa'} = 2k_s h_a$$
$$y = k_s D_{ab'}, \quad \theta = \sin^{-1}\left(\frac{d_{ab}}{D_{ab'}}\right)$$

$k_s = \sqrt{\omega\mu / \rho}$, where ρ is the soil resistivity

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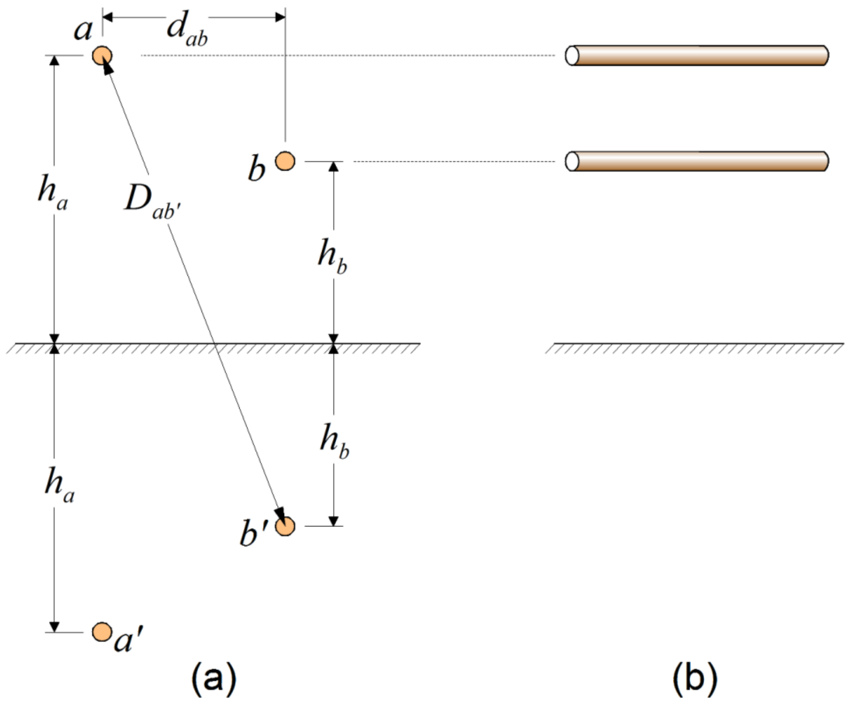

Figure 3.15: Two Parallel Power Conductors Above Soil

Note that the above equation provides the self and mutual series impedances of any conductor or any pair of conductors respectively. Specifically, the self-series impedance of conductor a is:

$$z_{a,series} = r_a + j\frac{\omega\mu}{2\pi}\ln\frac{D_{aa'}}{d} + \frac{\omega\mu}{\pi}(P_{aa} + jQ_{aa})$$

The mutual series impedance between conductors ***a*** and ***b*** is:

$$z_{ab,series} = j\frac{\omega\mu}{2\pi}\ln\frac{D_{ab'}}{d_{ab}} + \frac{\omega\mu}{\pi}(P_{ab} + jQ_{ab})$$

The above impedances are given in per unit length. Note that these equations can be repeated for any conductor and any pair of conductors of any complex arrangement of n conductors.

Equivalent Depth of Return Method: This method is obtained from the general solution (Carson [???]) presented earlier if only the first term of the infinite series is retained. The basic equations of this model can be stated with the aid of Figure 3.15 which illustrates two horizontal conductors above earth. The two conductors may be the two phases of a line, a phase conductor and a shield conductor, etc. The induced voltage on conductor ***a*** is expressed in terms of the equivalent depth of return, D_e, defined by

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$$D_e = 2160\sqrt{\frac{\rho}{f}}\ (feet) = 658.368\sqrt{\frac{\rho}{f}}\ (meters)$$

where ρ is the soil resistivity in Ohm-meters, and f is the electric current frequency in Hz. The induced voltage per unit length of conductor a is

$$\tilde{V}_a = \left(r_a + r_e + j\frac{\omega\mu}{2\pi}\ln\frac{D_e}{d_a} \right)\tilde{I}_a + \left(r_e + j\frac{\omega\mu}{2\pi}\ln\frac{D_e}{D_{ab}} \right)\tilde{I}_b$$

$$r_a = r_{dc}\frac{ka}{2}\frac{M_0(ka)}{M_1(ka)}\sin\left(\theta_1(ka) - \theta_0(ka) - \frac{\pi}{4} \right) ohms/meter$$

$$r_e = \frac{\omega\mu}{8},\ k = \sqrt{\omega\mu\sigma},\ \text{and}\ d_a = ae^{-\frac{\xi}{4}}$$

$$\text{with}\quad \xi = \frac{4}{ka}\frac{M_0(ka)}{M_1(ka)}\sin\left(\theta_0(ka) - \theta_1(ka) + \frac{3\pi}{4} \right)$$

a = radius of conductor ***a***
ω = angular frequency
μ = permeability of free space ($4\pi x10^{-7}$ H/m)
σ = conductivity of the conductor
M_0, θ_0 = modulus and phase of the modified Bessel function of first kind and zero order
M_1, θ_1 = modulus and phase of the modified Bessel function of first kind and first order.

Above equation provides the self and mutual series impedance of any conductor or any pair of conductors respectively. Specifically, the series self-series impedance of conductor a is:

$$z_{a,series} = r_a + r_e + j\frac{\omega\mu}{2\pi}\ln\frac{D_e}{d_a}$$

The mutual series impedance between conductors ***a*** and ***b*** is:

$$z_{ab,series} = r_e + j\frac{\omega\mu}{2\pi}\ln\frac{D_e}{D_{ab}}$$

The above method is called the equivalent depth of return method. It is also many times referred to as Carson's equation. This simplified formula is valid only for usual soil resistivities (20 to 500 Ohm.m) and for low frequencies such as the power frequency (50 or 60 Hz), and for usual overhead line configurations.

Interpretation of the Equivalent Depth of Return Method: A physical interpretation of the equivalent depth of return method can be provided as follows. The equivalent depth, D_e, defines the cross section of the soil under the line where the majority of the electric current returns. For example, consider the simple case of one conductor above earth carrying an electric current and the electric current returns through the earth. The return current is spread into the soil and most of the current returns to the source through a semi-circle with radius equal to the equivalent depth of return. The higher the frequency the smaller the radius will be. This interpretation applies to any electric current under a multi-conductor line in which case the return current through the earth will be the negative sum of all currents in the conductors of the line. A visualization of this interpretation is provided in Figure 3.16.

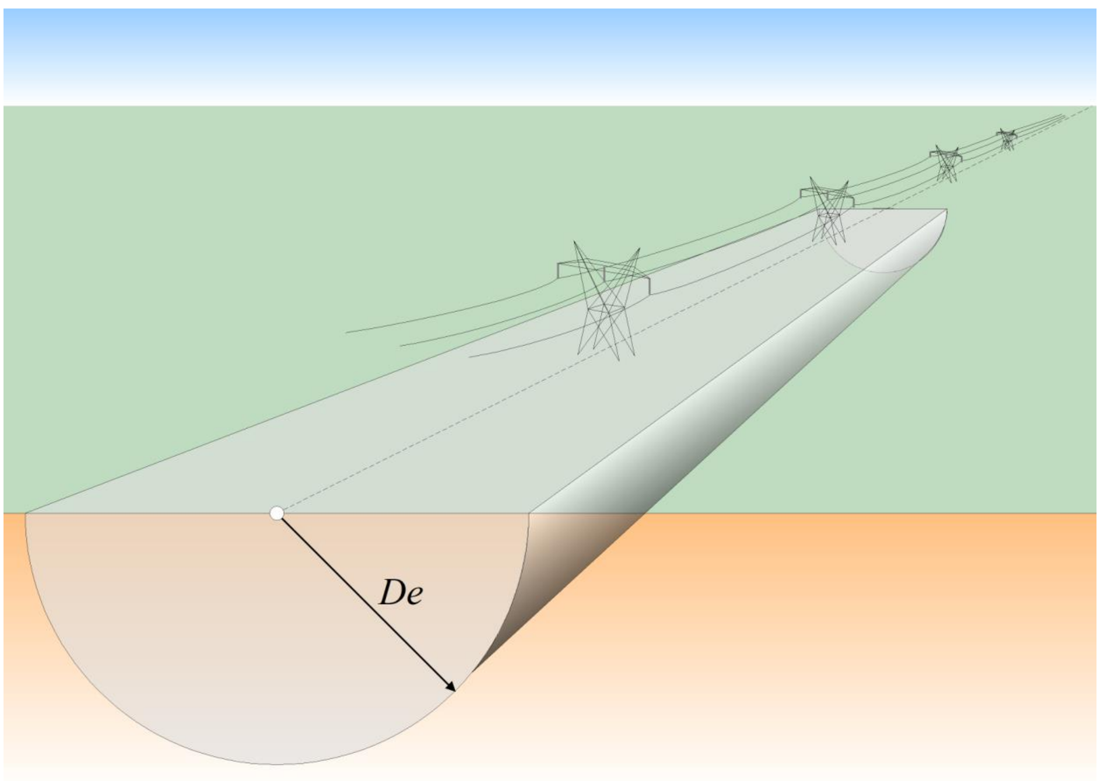

Figure 3.16: Interpretation of the Equivalent Depth of Return Method

Complex Depth of Return Method: Another method which is provided in closed form and it is remarkably accurate over a wide frequency range is the complex depth of return method. The basic equations of this model can be stated with the aid of Figure 3.15 which illustrates two horizontal conductors above earth. The two conductors may be the two phases of a line, a phase conductor and a shield conductor, etc. The induced voltage on conductor *a* is expressed in terms of the complex depth p [???] [???], defined by:

$$p = \frac{1.0}{\sqrt{j\omega\mu / \rho}}$$

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where ρ is the soil resistivity. The induced voltage per unit length of conductor a is

$$\tilde{V}_a = \left(r_a + j\frac{\omega\mu}{2\pi}\ln\frac{2(h_a+p)}{d} \right)\tilde{I}_a + \left(j\frac{\omega\mu}{2\pi}\ln\frac{\sqrt{(h_a+h_b+2p)^2+d_{ab}^2}}{\sqrt{(h_a-h_b)^2+d_{ab}^2}} \right)\tilde{I}_b$$

$$r_a = r_{dc}\frac{ka}{2}\frac{M_0(ka)}{M_1(ka)}\sin\left(\theta_1(ka)-\theta_0(ka)-\frac{\pi}{4}\right) ohms/meter$$

$$k=\sqrt{\omega\mu\sigma}\text{ , and } d_a = ae^{-\frac{\xi}{4}}$$

$$\text{with}\quad \xi = \frac{4}{ka}\frac{M_0(ka)}{M_1(ka)}\sin\left(\theta_0(ka)-\theta_1(ka)+\frac{3\pi}{4}\right)$$

Where:

h_a, h_b = heights of conductors a and b above ground (meters)
d_{ab} = horizontal separation between conductors ***a*** and ***b*** (meters)
a = radius of conductor ***a*** (meters)
ω = angular frequency (rad/s)
μ = permeability of free space ($4\pi x10^{-7}$H/m)
σ = conductor conductivity (S/m)
ρ = soil resistivity (Ωm)
M_0, θ_0 = modulus and phase of the modified Bessel function of first kind and zero order.
M_1, θ_1 = modulus and phase of the modified Bessel function of first kind and first order.

The above equation provides the self and mutual series impedance of any conductor or any pair of conductors respectively. Specifically, the series self-series impedance of conductor ***a*** is:

$$z_{a,series} = r_a + r_e + j\frac{\omega\mu}{2\pi}\ln\frac{2(h_a+p)}{d}$$

The mutual series impedance between conductors ***a*** and ***b*** is:

$$z_{ab,series} = r_e + j\frac{\omega\mu}{2\pi}\ln\frac{\sqrt{(h_a+h_b+2p)^2+d_{ab}^2}}{\sqrt{(h_a-h_b)^2+d_{ab}^2}}$$

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The above method is called the *complex depth of return method* which is a closed-form approximation to Carson's solution and was suggested by Semlyen and Deri [???]. This closed-form solution yields a remarkably close agreement with the exact Carson's solution in a wide range of frequencies (0 to 10 MHz) for typical overhead line configurations.

Summary of the Three Methods: Note that each one of the three presented methods provide the self and mutual impedance of two parallel conductors above earth. These results can be easily generalized to an n-conductor configuration above soil by considering two conductors at a time. Specifically, the series impedance of an n-conductor power line is provided by

$$Z = R + j\omega L = \begin{bmatrix} R_{11}+jX_{11} & R_{12}+jX_{12} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ R_{n1}+jX_{n1} & R_{n2}+jX_{n2} & \cdots & R_{nn}+jX_{nn} \end{bmatrix}$$

The elements of the above matrix can be computed with any of the three methods presented.

Example E3.2: Consider the three-phase electric power line of Figure E3.2. The phase conductors are ACSR, 556,500 cm, 26 strands. The line does not have an overhead ground wire. The soil resistivity ρ is 75 Ωm. Compute the resistance and inductance matrices of this line using: (a) The equivalent depth of return, (b) The complex depth of return method.

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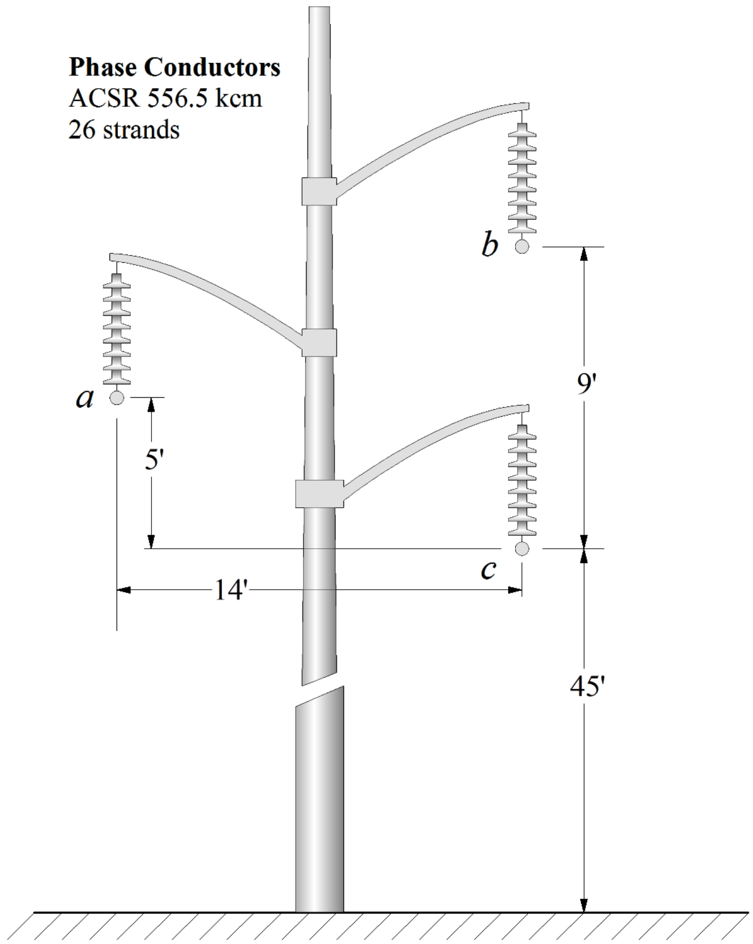

Figure E3.2

Solution: (a) The series impedance of the line using the equivalent depth of return method is:

$$R + j\omega L = \begin{bmatrix} r + r_e & r_e & r_e \\ r_e & r + r_e & r_e \\ r_e & r_e & r + r_e \end{bmatrix} + j\frac{\omega\mu}{2\pi}\begin{bmatrix} \ln\frac{D_e}{d_{aa}} & \ln\frac{D_e}{d_{ab}} & \ln\frac{D_e}{d_{ac}} \\ \ln\frac{D_e}{d_{ba}} & \ln\frac{D_e}{d_{bb}} & \ln\frac{D_e}{d_{bc}} \\ \ln\frac{D_e}{d_{ca}} & \ln\frac{D_e}{d_{cb}} & \ln\frac{D_e}{d_{cc}} \end{bmatrix}$$

From ACSR conductor tables we obtain the following conductor parameters:

- Conductor Resistance at 60 Hz, r = 0.1611 Ω/mile = 0.0001 Ω/m.
- Conductor Geometric Mean Radius: 0.0315 feet.

Furthermore,

$$D_e = 2{,}160\sqrt{\frac{75}{60}} = 2{,}415 \ ft$$
$$d_{ab} = d_{ba} = 14.56 \ ft$$
$$d_{ac} = d_{ca} = 14.87 \ ft$$
$$d_{bc} = d_{cb} = 9.0 \ ft$$
$$d_{aa} = d_{bb} = d_{cc} = 0.0315 \ ft$$

Upon substitution into the above impedance matrix formula:

$$R + j\omega L = 10^{-3}\begin{bmatrix} 0.159 & 0.059 & 0.059 \\ 0.059 & 0.159 & 0.059 \\ 0.059 & 0.059 & 0.159 \end{bmatrix} + j10^{-3}\begin{bmatrix} 0.8478 & 0.3853 & 0.3838 \\ 0.3853 & 0.8478 & 0.4215 \\ 0.3838 & 0.4215 & 0.8478 \end{bmatrix} \quad Ohms/meter$$

(b) The series impedance of the line using the complex depth of return method is:

$$R + j\omega L = \begin{bmatrix} z_{a,series} & z_{ab,series} & z_{ac,series} \\ z_{ba,series} & z_{\mathrm{b},series} & z_{bc,series} \\ z_{ca,series} & z_{cb,series} & z_{\mathrm{c},series} \end{bmatrix}$$

Where:

$$z_{\mathrm{i},series} = r_i + j\frac{\omega\mu}{2\pi}\ln\frac{2(h_i + p)}{d}, \; i = a, b, c \text{ , and}$$

$$z_{ik,series} = j\frac{\omega\mu}{2\pi}\ln\frac{\sqrt{(h_i + h_k + 2p)^2 + d_{ik}^2}}{\sqrt{(h_i - h_k)^2 + d_{ik}^2}}, \quad ik = ab, bc, ca$$

$$\text{where:} \quad p = \frac{1.0}{\sqrt{j\omega\mu/\rho}}$$

Upon substitution one obtains:

$$p = 281.349 - j281.349\, m$$

and:

$$R + j\omega L = 10^{-3}\begin{bmatrix} 0.157 & 0.057 & 0.057 \\ 0.057 & 0.157 & 0.057 \\ 0.057 & 0.057 & 0.158 \end{bmatrix} + j10^{-3}\begin{bmatrix} 0.8559 & 0.3906 & 0.3876 \\ 0.3906 & 0.8561 & 0.4034 \\ 0.3876 & 0.4034 & 0.8557 \end{bmatrix} \quad Ohms/meter$$

Comparing the results of (a) and (b) it appears that they are remarkably close.

3.2.3 Capacitance

In this section we discuss methods by which the capacitance of a transmission line can be computed. For this purpose we employ an approach analogous to the one for computing the inductive reactance of a transmission line. Recall that for the computation of the inductive reactance, the magnetic field around the transmission line was examined. For the computation of the line capacitance, the electric field around the line will be examined. The source of this electric field is electric charge, which is deposited on the surface of the line conductors. The analysis of the electric field results in a model relating the electric charge and the conductor voltage. The time derivative of the total electric charge on the surface of the conductors is by definition the capacitive current (or the charging current) of the line. Utilizing this definition, the model can be transformed into a relationship between the line voltage and the capacitive current. The line capacitance can be extracted from this model.

This general approach will be utilized to introduce the analysis of capacitive phenomena in lines in a step-by-step procedure. Specifically, first the simplest case of a single circular conductor will be examined to establish the basic equations. Then the analysis will be extended to two parallel conductors and the general n-conductor line configuration.

3.2.3.1 Basic Electric Field Equations around a Conductor

Consider the simple case of one circular infinitely long conductor. We shall assume that the conductor is electrically charged and we shall seek the relationship between the electric charge and the conductor voltage. Specifically, assume that the conductor is charged with electric charge q (coulombs per meter). Because of symmetry, the electric charge will be uniformly

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distributed on the conductor surface. The electric charge generates an electric field around the conductor. Because of symmetry, the electric field intensity E will be radially directed and the magnitude will depend only on the distance of the point of observation from the axis of the conductor, as illustrated in Figure 3.17:

$$\vec{E} = E(r)\vec{a}_r \tag{3.13}$$

Where $\vec{a}_r$ is a unit vector in the radial direction r.

Consider a cylinder of length ℓ and circular bases of radius r. The axis of the cylinder is coincident with the axis of the conductor, as it is illustrated Figure 3.17. Let S be the surface of the cylinder and V its volume. Application of Gauss's law yields:

$$\iiint_V \rho dv = \iint_S \vec{D}.d\vec{s} = \iint_S \varepsilon \vec{E}.d\vec{s} \tag{3.14}$$

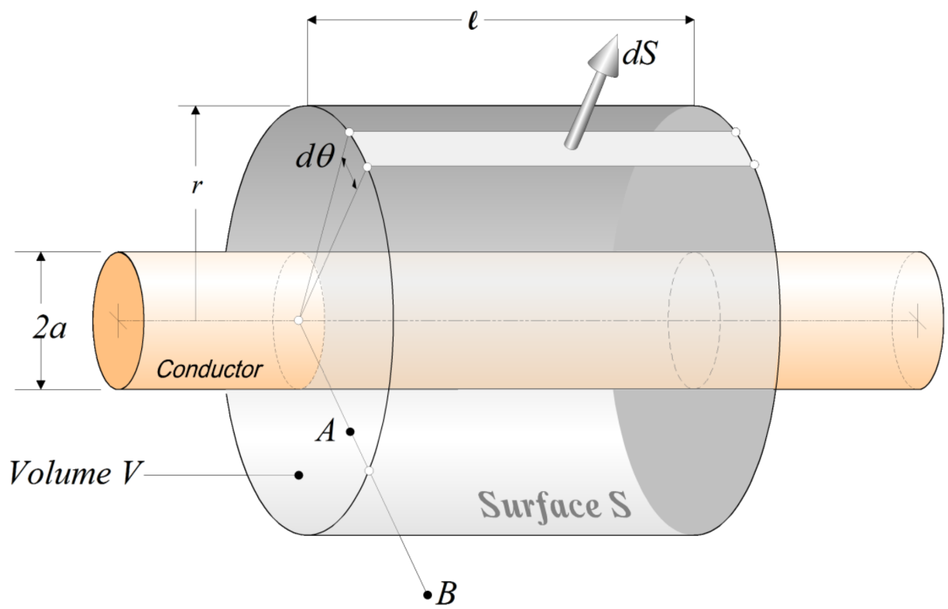

Figure 3.17: An Infinitely Long Circular Conductor

where

ρ = electric charge density, C/m^3

$\vec{E}$ = electric field intensity

$\vec{D}$ = electric field density

dv = infinitesimal volume

$d\vec{s}$ = infinitesimal surface area vector

The volume integral of the electric charge density inside the volume of the cylinder equals the total electric charge enclosed in the volume. It can be immediately computed by observing that electric charge exists only on the conductor surface at a density of q coulombs per meter. Thus

$$\iiint_V \rho dv = q\ell$$

The surface integral on the right-hand side of Equation (3.14) is computed as follows:

$$\iint_S \vec{D}.d\vec{s} = \iint_{S_1} \vec{D}.d\vec{s} + \iint_{S_2} \vec{D}.d\vec{s} + \iint_{S_3} \vec{D}.d\vec{s}$$

where S_1, S_2 are the bases of the cylinder and S_3 is the side surface of the cylinder. Note that because the electric field is radially directed, the contributions of the bases of the cylinder will vanish, that is,

$$\iint_{S_1} \vec{D}.d\vec{s} = \iint_{S_2} \vec{D}.d\vec{s} = 0.0$$

As has been discussed, the magnitude of the electric field intensity $\vec{E}$ and therefore $\vec{D}$ is a function of the radial distance r only. Thus on the surface S_3, the magnitude of the electric field density, D(r), is constant. In addition, the vector $\vec{D}$ is perpendicular to the surface S_3 and thus parallel to $d\vec{s}$. Thus

$$\iint_{S_3} \vec{D}.d\vec{s} = 2\pi r\ell D(r)$$

Substitution into Eq. (4.2) yields

$$q\ell = 2\pi r\ell D(r) = 2\pi r\ell \varepsilon E(r)$$

In above equation we used the constitutive relationship: D(r) = εE(r). Solution of above equation for E(r) yields:

$$E(r) = \frac{q}{2\pi\varepsilon r} \qquad (3.15)$$

The electric field inside the conductor is zero.

The computed electric field intensity provides the basis for computation of the potential difference between any two points A and B. This difference is the voltage V_{AB} between point A and B, defined by:

$$V_{AB} = \Phi(A) - \Phi(B) = \int_{C_{A\to B}} \vec{E}(r).d\vec{\ell}$$

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The value of above integral depends only on points A and B (the reader is encouraged to prove it). Evaluation of the integral yields:

$$V_{AB} = \int_{C_{A \to B}} \vec{E}(r).d\vec{\ell} = \frac{q}{2\pi\varepsilon} \ln \frac{d_B}{d_A} \tag{3.16}$$

where: d_A *and* d_B are the distances of points A and B respectively from the axis of the conductor.

Equation (3.16) relates the electric charge on the conductor to the potential difference between two points located at radial distances d_A *and* d_B, respectively, from the axis of the conductor. Equation (3.16) is the basic equation utilized in the analysis of transmission line capacitance.

3.2.3.2. Capacitive Equations of a Multi-Conductor Line

Consider a configuration of n conductors which are parallel and infinitely long. The conductor cross section is circular. Figure 3.18 shows a cross section of the configuration. Assume that electric charge $q_i(t)$ per unit length has been accumulated on the surface of conductor i which is uniformly distributed over the surface of the conductor. As a first step, we consider the potential of conductor i with respect to an arbitrarily selected point of reference X which is illustrated in Figure 3.18. For this purpose the principle of superposition and the results of section 3.2.3.2 are employed to yield

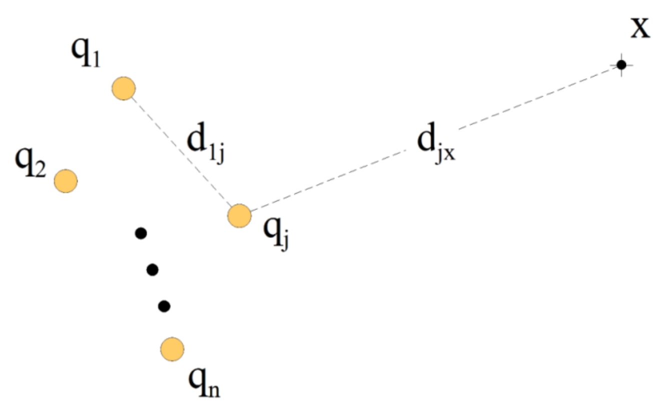

Figure 3.18: General Configuration of n-Parallel Conductors

$$V_{ix}(t) = \Phi_i(t) - \Phi_x(t) = \frac{1}{2\pi\varepsilon} \sum_{j=1}^{n} q_j(t) \ln \frac{d_{jx}}{d_{ij}} \tag{3.17}$$

where

d_{ij} = distance between the axes of conductors i and j

d_{jx} = distance between the axis of conductor j and point X

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